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Classical ultra-relativistic scattering in ADD

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Abstract: The classical differential cross-section is calculated for high-energy small-angle gravitational scattering in the factorizable model with toroidal extra dimensions. The three main features of the classical computation are: (a) It involves summation over the infinite Kaluza-Klein towers but, contrary to the Born amplitude, it is finite with no need of an ultraviolet cutoff. (b) It is shown to correspond to the non-perturbative saddle-point approximation of the eikonal amplitude, obtained by the summation of an infinite number of ladder graphs of the quantum theory. (c) In the absence of extra dimensions it reproduces all previously known results.

Contents

1. Introduction

The search for large extra dimensions (LED), especially in view of the forthcoming experiments at LHC, constitutes a very exciting direction in high energy physics. After precursor ideas about the Universe as a topological defect in higher-dimensional space-time [1] and the proposal of TeV-scale internal dimensions related to supersymmetry breaking in string theory [2], large extra dimensions were discussed in several contexts. Among the conceptually and technically simplest is the ADD picture [3], according to which the standard model particles live in a four-dimensional space (the brane), embedded in a D-dimensional bulk inhabited only by gravity, and with the extra $\delta = D - 4$ dimensions compactified on a torus. The D-dimensional Planck mass M_* is supposed to lie in the TeV region and the LED have submillimeter size. In this scenario there is an infinite tower of massive Kaluza-Klein (KK) gravitons [4] whose existence may be detected at present and future colliders [5], either as missing energy in collisions due to emission of KK gravitons (which are weakly interacting after being created), or via processes which would otherwise be impossible or very much suppressed.

Quantum collision processes with exchange of virtual KK gravitons are a very useful tool to test the model [5]. Unfortunately, tree diagrams containing the propagator of the KK tower diverge in the ultraviolet (UV) due to an infinite sum over the KK graviton masses [4]. The divergence appears already at *tree* level and is due to the emission of infinite momenta into the compactified dimensions. Several proposals were made in order to cure this problem. One was to introduce a UV cutoff [4, 6], another to take into account brane oscillations and the associated Nambu-Goldstone modes [7], another yet to introduce brane thickness [8], or finally, to sum up ladder diagrams within the eikonal approximation [9]. However, none of them solves the problem completely. The UV cutoff at Planckian energies while it certainly does exist, does not look very natural in this context. It would mean that we are able to probe the Planckian regime using (relatively)

low energy processes. Appealing to features associated with the *physical brane*, such as tension and thickness, introduces additional elements into the ADD model, which seem to spoil its overall consistency. Indeed, for a brane of non-zero tension one can not use any more the flat Minkowski background. Instead, one has to deal with solutions of the Einstein equations, which are non-trivial even in the absence of matter on the brane. In the case of codimension one, such a solution is well-known and is the basis of the Randall-Sundrum II model [10], with a different graviton spectrum. Finally, the eikonal calculation amounts to using the Fourier-transform of the Born amplitude and thus, it also suffers from ambiguities associated with the divergence of the latter. Indeed, the evaluation of the eikonal phase gives a finite result if a certain order of integration is used in calculating the Fourier integral, while it is divergent if one merely changes the order of integration.

On the other hand, it has been argued that at ultrahigh energies, particle scattering in four dimensions not only becomes dominated by gravity, but in addition it involves only *classical* gravitational dynamics [11, 12, 13, 14]. Indeed, quantum gravity effects should not, by definition, be important in the classical limit $\hbar \to 0$. This, in terms of the two relevant lengths, i.e. the Planck length $l_{\rm Pl} = (\hbar G_4/c^3)^{1/2} = \hbar/M_{\rm Pl}c$ and the gravitational radius associated with the energy of the collision $r_g = G_4\sqrt{s}/c^4$, implies that $r_g \gg l_{\rm Pl}$, which is equivalent to the condition $\sqrt{s} \gg M_{\rm Pl}c^2$ of transplanckian energies. Thus, to study the scattering of two point-like particles with Planck energy, 't Hooft used a shock wave approximation for the field of the moving particle and obtained a result similar to the Veneziano amplitude. Later on, it was shown that in four dimensional quantum gravity [15], as well as in string theory [13], the eikonal approximation (free in both cases of the ambiguities mentioned above) reproduced the result of 't Hooft.

The Planck length l_* and the gravitational radius r_g^* in the D-dimensional ADD model are, correspondingly

$$l_* = \left(\frac{\hbar G_D}{c^3}\right)^{\frac{1}{\delta+2}} \sim \frac{\hbar}{M_* c}, \quad r_g^* = \left(\frac{G_D\sqrt{s}}{c^4}\right)^{\frac{1}{\delta+1}}, \tag{1.1}$$

where M_* is the TeV-scale mass parameter. The above reasoning remains essentially the same and shows that in the transplanckian regime $\sqrt{s} \gg M_* c^2$ scattering is also classical, at least for some range of momentum transfer. Moreover, the particularly exciting proposals of black hole creation at LHC or in cosmic rays [16] [17] were based on a purely classical picture.

Although the eikonal approximation of particle scattering in ADD has been discussed by a number of authors [9], [8], no classical calculation of the cross-section was found in the literature. The purpose of the present paper is to fill in this gap, and in addition to provide an independent check of the validity of the eikonal approach, whose applicability in the context of ADD is not yet rigorously proven. Indeed, it is hereby demonstrated explicitly that the classical theory reproduces the saddle-point result of the eikonal approximation and is essentially non-perturbative in the quantum sense. Thus, classical theory gives non-trivial results in ADD, in contrast to four dimensional gravity where the classical elastic scattering cross-section coincides with the Born approximation and *is* perturbative. Finally, it should be pointed out that the reason the classical cross-section is finite, even

though it too involves an infinite summation over massive KK modes, is because in the classical calculation the potentially divergent integrals contain oscillating factors, which effectively cutoff the modes that cannot be excited by the source. This applies to all kinematical regimes and, in particular, is analogous to the situation in the classical derivation of Newton's law [18] [19].

2. The setup

Consider the Fierz-Pauli lagrangian in D-dimensional Minkowski space, with $\delta \equiv D-4$ of its spatial dimensions being a torus T^δ with equal radii R

$$\mathcal{L} = -\frac{1}{4}h^{MN}\Box h_{MN} + \frac{1}{4}h\Box h - \frac{1}{2}h^{MN}\partial_M\partial_N h + \frac{1}{2}h^{MN}\partial_M\partial_P h^P_N - \frac{\varkappa_D}{2}h^{MN}T_{MN}, \quad (2.1)$$

where $M, N, .. = 0, 1, 2, ..., D-1$. The Minkowski metric is $\eta_{MN} = \mathrm{diag}(1, -1, -1, ...)$ and $\Box \equiv \eta^{MN}\partial_M\partial_N$. The gravitational field h_{MN} is coupled to a conserved matter stress-tensor T_{MN} $(\partial_N T^{MN} = 0)$. All fields $(h \equiv \eta^{MN}h_{MN} \equiv h^M_M)$ are functions of

$$x^M = (x^\mu, y^i), \quad \mu = 0, \ldots, 3, \quad i = 1, \ldots \delta, \quad (2.2)$$

and are periodic under the translations $y_j \to y_j + 2\pi R, \quad j = 1, \ldots, \delta$. Thus, for instance

$$h_{MN}(x^P) = \sum_{n_1=-\infty}^{+\infty} \cdots \sum_{n_\delta=-\infty}^{+\infty} \frac{h^n_{MN}(x)}{\sqrt{V}} \exp\left(i\frac{n_i y^i}{R}\right), \quad (2.3)$$

where $V = (2\pi R)^\delta$ is the volume of the torus. In what follows we abbreviate the sum over the KK modes as $\sum_n$ and denote the momenta transversal to the brane as $p^i{}_T = n^i/R$. The tensor $h_{MN}(x^P)$ has been split into an infinite sum of four-dimensional KK modes $h^n_{MN}(x^\mu)$ with $(\mathrm{mass})^2$ equal to p_T^2.

In the harmonic gauge $\partial_N h^{MN} = \frac{1}{2}\partial^M h$ the Einstein equations for h_{MN} read:

$$\Box h_{MN} = -\varkappa_D\left(T_{MN} - \frac{T}{\delta+2}\eta_{MN}\right) \equiv -J_{MN}, \quad T = \eta^{MN}T_{MN}. \quad (2.4)$$

According to the ADD scenario it is assumed that the matter stress-tensor is localized on the brane, located at $\mathbf{y} = 0$, and carries only four-dimensional indices:

$$T_{MN}(x^P) = \eta^\mu_M \eta^\nu_N T_{\mu\nu}(x)\delta^\delta(\mathbf{y}). \quad (2.5)$$

It is worth noting, however, that this assumption is consistent with Einstein's equations only at the linearized level, with matter dynamics governed by the non-gravitational forces alone. Given (2.5) it is consistent to set the graviphotons $h_{i\mu}$ and the non-diagonal part of the scalar matrix h_{ij} to zero. Finally, the diagonal components of h_{ij} are all equal and generated by the trace of the energy-momentum tensor:

$$\Box h_{ij} = -\varkappa_D \frac{T}{\delta+2}\delta_{ij}. \quad (2.6)$$

Their zero modes $n^i = 0$ are the so called *radions*, which describe deformations of the torus caused by the presence of matter on the brane. The massive components $n^i \neq 0$ of h^n_{MN} could be rearranged into the massive four-dimensional graviton and massive scalars in the usual Higgs mechanism language [20], [4], but this is not necessary for the present discussion.

The D-dimensional Planck mass M_* is defined by $\varkappa_D^2 = 16\pi/M_*^{\delta+2} \equiv 16\pi G_D$ and is related to the four-dimensional one $M_{\rm Pl}$ via $M_{\rm Pl}^2 = M_*^{\delta+2}V$.

The retarded Green's function of the D'Alembert equation satisfies

$$\Box G_D(x, x', \mathbf{y} - \mathbf{y}') = -\delta^4(x - x')\delta^\delta(\mathbf{y} - \mathbf{y}'). \tag{2.7}$$

Its Fourier transform reads:

$$G_D(x - x', \mathbf{y} - \mathbf{y}') = \frac{1}{(2\pi)^4 V} \int d^4p\, \mathrm{e}^{-ip\cdot(x-x')} \sum_n \frac{\mathrm{e}^{i\mathbf{p}_T\cdot(\mathbf{y}-\mathbf{y}')}}{p^2 - p_T^2 + i\epsilon p^0}. \tag{2.8}$$

The solution of (2.4) with the source localized on the brane

$$J_{MN}(x, \mathbf{y}) = J_{MN}(x)\delta^\delta(\mathbf{y}), \tag{2.9}$$

is

$$\begin{aligned} h_{MN}(x, \mathbf{y}) &= \int G_D(x - x', \mathbf{y} - \mathbf{y}')J_{MN}(x', \mathbf{y}')d^4x' d\mathbf{y}' \\ &= \int \frac{d^4p\, e^{-ip\cdot(x-x')}}{(2\pi)^4 V} \sum_n \frac{\mathrm{e}^{i\mathbf{p}_T\cdot\mathbf{y}}}{p^2 - p_T^2 + i\epsilon p^0}\, J_{MN}(x')\, d^4x'. \end{aligned} \tag{2.10}$$

Its restriction to the brane

$$h_{MN}(x) \equiv h_{MN}(x, \mathbf{y})\Big|_{\mathbf{y}=0} \tag{2.11}$$

can be rewritten using the amputated propagator

$$h_{MN}(x) = \int G_4(x - x')J_{MN}(x')d^4x', \quad G_4(x - x') = G_D(x - x', \mathbf{y} - \mathbf{y}')\Big|_{\mathbf{y}=\mathbf{y}'=0}. \tag{2.12}$$

The momentum space four-dimensional retarded propagator thus reads

$$G_4(p) = \frac{1}{V} \sum_n \frac{1}{p^2 - p_T^2 + i\epsilon p^0}. \tag{2.13}$$

Equivalently, taking the four-dimensional Fourier transform, defined by $\Psi(k) = \int \Psi(x)\mathrm{e}^{ik\cdot x}d^4x$ and $\Psi(x) = \int \Psi(k)\mathrm{e}^{-ik\cdot x}d^4k/(2\pi)^4$ of the above equations, the retarded solution of Eq. (2.4) becomes

$$h_{MN}(k) = \frac{\varkappa_D}{V} \sum_n \frac{T_{MN}(k) - \frac{1}{\delta+2}\eta_{MN}T(k)}{k^2 - p_T^2 + i\epsilon k^0}. \tag{2.14}$$

3. The ultra-relativistic elastic scattering cross-section

Consider next the small angle ultrarelativistic scattering of two particles on the brane, with masses m and m' respectively, interacting via D-dimensional gravity. Using for notational simplicity the same parameter τ for both trajectories, a convenient way to describe them is

$$z^\mu(\tau) = z_0^\mu + \frac{p^\mu}{m}\tau + \delta z^\mu(\tau), \quad z'^\mu(\tau) = z_0'^\mu + \frac{p'^\mu}{m'}\tau + \delta z'^\mu(\tau), \tag{3.1}$$

with $\delta z, \delta z'$ treated perturbatively. The asymptotic values of their momenta are

$$\begin{aligned} P^\mu &= m \lim_{\tau\to-\infty} \dot{z}^\mu(\tau) = p^\mu + m \lim_{\tau\to-\infty} \delta\dot{z}^\mu(\tau), \quad P'^\mu = m' \lim_{\tau\to-\infty} \dot{z}'^\mu(\tau) = p'^\mu + m' \lim_{\tau\to-\infty} \delta\dot{z}'^\mu(\tau), \\ \bar{P}^\mu &= m \lim_{\tau\to\infty} \dot{z}^\mu(\tau) = p^\mu + m \lim_{\tau\to\infty} \delta\dot{z}^\mu(\tau), \quad \bar{P}'^\mu = m' \lim_{\tau\to\infty} \dot{z}'^\mu(\tau) = p'^\mu + m' \lim_{\tau\to\infty} \delta\dot{z}'^\mu(\tau), \end{aligned} \tag{3.2}$$

while momentum conservation implies $P^\mu + P'^\mu = \bar{P}^\mu + \bar{P}'^\mu$. The Mandelstam variables s, t are

$$s = (P+P')^2, \quad t = q^2 = (\bar{P} - P)^2 = m^2 \left(\lim_{\tau\to\infty} \delta\dot{z}^\mu(\tau) - \lim_{\tau\to-\infty} \delta\dot{z}^\mu(\tau) \right)^2, \tag{3.3}$$

and we consider the ultrarelativistic regime and the small-angle approximation

$$s \gg m^2,\ m'^2,\ |t|, \tag{3.4}$$

in which case $s = (p+p')^2$.

Since the momentum transfer q^μ depends only on the deviation δz^μ of one of the particles, it will be convenient to work in the rest frame of m' before collision. In that frame

$$p'^\mu = m'(1,0,0,0), \quad p^\mu = m\gamma(1,0,0,v), \quad \gamma = \frac{1}{\sqrt{1-v^2}}, \tag{3.5}$$

and with no loss of generality one may set in addition

$$z_0'^\mu = 0, \quad z_0^\mu = b^\mu = (0,b,0,0), \tag{3.6}$$

where b is the impact parameter.

The particles equations of motion following from the (2.1) are D-dimensional geodesic equations in the metric

$$g_{MN} = \eta_{MN} + \varkappa_D h_{MN}, \tag{3.7}$$

but it is easy to show that the particle moving on the brane in zero order in $\varkappa_D$ will remain on the brane. So the matter stress-tensor of the two particle system is four-dimensional:

$$T^{\mu\nu} = -\int \left[m\dot{z}^\mu \dot{z}^\nu \delta^4(x - z(\tau)) + m'\dot{z}'^\mu \dot{z}'^\nu \delta^4(x - z'(\tau)) \right] d\tau. \tag{3.8}$$

Leaving aside the classical mass renormalization needed to take into account the gravitational self-action, we have to take as $h_{\mu\nu}$ in the equation for the particle m the retarded

field of the partner particle m'. In the small-angle approximation it is assumed that the deviation from the unperturbed rectilinear motion is small, and we get perturbatively

$$\Pi^{\mu\nu}\delta\ddot{z}_\nu = \varkappa_D \Pi^{\mu\nu}\left(h_{\nu\lambda,\rho} - \frac{1}{2}h_{\lambda\rho,\nu}\right)\frac{p^\lambda p^\rho}{m^2}, \tag{3.9}$$

where $\Pi^{\mu\nu}$ is the projector onto the space transverse to the momentum p^μ

$$\Pi^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}. \tag{3.10}$$

The gravitational field $h_{\mu\nu}$ of the particle m' is given by (2.14), where we have to substitute the Fourier transform of the second term of the stress-tensor (3.8)

$$h_{\mu\nu}(k) = \frac{2\pi\varkappa_D}{V}\sum_n \frac{\delta(k\cdot p')}{k^2 - p_T^2 + i\epsilon k^0}\left(p'_\mu p'_\nu - \frac{m'^2\eta_{\mu\nu}}{\delta+2}\right). \tag{3.11}$$

Using the reparametrization invariance of the particle trajectories one may choose τ in such a way that $p^\mu\delta\dot{z}_\mu = 0$ for both particles. In this gauge the solution of (3.9) becomes

$$\delta z^\mu(\tau) = \frac{i\varkappa_D^2}{(2\pi)^3 V}\int\sum_n \frac{\delta(k\cdot p')\mathrm{e}^{ik\cdot b - ik\cdot p\tau/m}}{(k\cdot p)(k^2 - p_T^2 + i\epsilon k^0)}Q^\mu d^4k. \tag{3.12}$$

The vector Q^μ has the form

$$Q^\mu = Ak^\mu + Bp^\mu + Cp'^\mu, \tag{3.13}$$

where

$$A = \frac{1}{2k\cdot p}\left((p\cdot p')^2 - \frac{m^2 m'^2}{\delta+2}\right), \quad B = \frac{1}{2m^2}\left((p\cdot p')^2 + \frac{m^2m'^2}{\delta+2}\right), \quad C = -p\cdot p'. \tag{3.14}$$

Upon differentiation with respect to τ and integration over k^0, using that in the chosen Lorentz frame $\delta(k\cdot p') = \delta(m'k^0)$, one obtains

$$\delta\dot{z}^\mu(\tau) = \frac{2}{\pi^2 m' m M_*^{2+\delta} V}\int\sum_n \frac{\mathrm{e}^{ik\cdot b - ik\cdot p\tau/m}}{\mathbf{k}^2 + p_T^2}Q^\mu\bigg|_{k^0=0} d^3k. \tag{3.15}$$

To calculate the momentum transfer in (3.3) we need the asymptotic values of $\delta\dot{z}^\mu$ as $\tau\to\pm\infty$, which one computes next. In the chosen Lorentz frame the exponent in (3.15) is

$$k\cdot b - k\cdot p\tau/m = k_x b - k_z v\gamma\tau. \tag{3.16}$$

(a) Start with the integral over k_z in (3.15). Define

$$I^\mu(\tau) \equiv \int_{-\infty}^{\infty}\frac{Q^\mu \mathrm{e}^{-ik_z\gamma v\tau}}{k_z^2 + \varkappa^2}dk_z, \quad \varkappa^2 = k_x^2 + k_y^2 + p_T^2. \tag{3.17}$$

The terms B and C, as well as the one proportional to Ak_z vanish in the limit $\tau\to\pm\infty$, because

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-ik_z\gamma v\tau}}{k_z^2+\varkappa^2}dk_z = \frac{\pi}{\varkappa}\mathrm{e}^{-\varkappa\gamma v|\tau|} \to 0\ , \ \text{as}\ \ \tau\to\pm\infty. \tag{3.18}$$

The term proportional to Ak_0 is zero, because it is evaluated at $k_0 = 0$. The y component vanishes by parity. So, the only component left is the one proportional to Ak_x. In this case, parity implies that only the sine part of the exponential contributes. Use

$$\lim_{\tau\to\pm\infty} \frac{\sin(k_z\gamma v\tau)}{k_z\gamma v} = \pm\pi\delta(k_z\gamma v), \tag{3.19}$$

to perform the k_z integration. Then, insert A from (3.14) to obtain

$$\lim_{\tau\to\pm\infty} I^x(\tau) = \pm\frac{\pi k_x}{2\gamma v m\varkappa^2}\left((p\cdot p')^2 - \frac{m^2m'^2}{\delta+2}\right). \tag{3.20}$$

(b) Next, denote by $K^2 = k_y^2 + p_T^2$ and perform the integral over k_x

$$\int_{-\infty}^{\infty} \frac{k_x \mathrm{e}^{ik_xb}}{k_x^2+K^2} dk_x = \pi\mathrm{e}^{-Kb}. \tag{3.21}$$

(c) Insert this into (3.15) and replace the sum over KK modes by a continuous integration

$$\sum_n \to \frac{VS_{\delta-1}}{(2\pi)^\delta}\int_0^\infty p_T^{\delta-1}dp_T, \quad S_{\delta-1} = \frac{2\pi^{\delta/2}}{\Gamma(\delta/2)}, \tag{3.22}$$

to obtain

$$\delta\dot{z}^x(\pm\infty) = \pm\frac{1}{2^{2+\delta}\pi^{\delta/2}m'm^2\gamma v\Gamma(\delta/2)M_*^{2+\delta}}\left((p\cdot p')^2 - \frac{m^2m'^2}{\delta+2}\right)\int_{-\infty}^{\infty} dk_y \int_0^\infty \mathrm{e}^{-Kb}\, p_T^{\delta-1}dp_T. \tag{3.23}$$

(d) To perform the remaining integrations pass to polar coordinates $p_T = K\cos\alpha$, $k_y = K\sin\alpha$, $p_T^{\delta-1}dk_ydp_T = K^\delta dK\cos^{\delta-1}\alpha d\alpha$, and integrate over K from zero to infinity and over α from $-\pi/2$ to $\pi/2$. The asymptotic values of the velocity of m in the field of m' are then:

$$\delta\dot{z}^x(\pm\infty) = \pm\frac{2\Gamma(\delta/2+1)}{\pi^{\delta/2}b^{\delta+1}m'm^2\gamma vM_*^{2+\delta}}\left((p\cdot p')^2 - \frac{m^2m'^2}{\delta+2}\right). \tag{3.24}$$

Inserting this into (3.3) we find for the square of the momentum transfer

$$-t = 2^4\Gamma^2(\delta/2+1)\frac{m^2m'^2}{\pi^\delta\gamma^2v^2M_*^2}\frac{1}{(M_*b)^{2(\delta+1)}}\left(\frac{(p\cdot p')^2}{m^2m'^2} - \frac{1}{\delta+2}\right)^2. \tag{3.25}$$

In the ultrarelativistic limit $\gamma \gg 1, v \simeq 1$, one has $s = 2p\cdot p' \gg mm'$ and the above expression simplifies to:

$$-t = \frac{2^2\Gamma^2(\delta/2+1)}{\pi^\delta(M_*b)^{2(\delta+1)}}\frac{s^2}{M_*^2}. \tag{3.26}$$

Finally, the differential cross-section, defined as usual by $d\sigma = 2\pi b db$, is

$$\frac{d\sigma}{dt} = \frac{1}{(\delta+1)(-M_*^2t)}\left(-4\pi\Gamma^2(1+\delta/2)\frac{s^2}{M_*^2t}\right)^{1/(\delta+1)}. \tag{3.27}$$

In particular, for $\delta = 0$

$$\frac{d\sigma}{dt} = \frac{4\pi G_4^2s^2}{t^2}, \tag{3.28}$$

which coincides with the well-known formula for small angle scattering of General Relativity [21].

The scattering angle θ is given by $\tan\theta = \sqrt{-t}/m\gamma v$, thus, small scattering angles mean $|t| \ll m^2\gamma^2 v^2$, which for ultrarelativistic velocities gives the range of validity of our approximation

$$(M_* b)^{\delta+1} \gg \frac{m'}{M_*}. \tag{3.29}$$

4. Relation to the eikonal approximation

Consider the elastic scattering of massive scalar particles on the brane in the high-energy limit $s \gg m^2$. The Born amplitude contains the t-channel propagator involving the sum over Kaluza-Klein modes. Passing to the continuous integration over the momentum of gravitons $\mathbf{p}_T$ in extra dimensions we have [4]:

$$\mathcal{M}_{\rm Born}(s,t) = \frac{s^2 \varkappa_D^2}{2(2\pi)^\delta} \int \frac{d^\delta p_T}{-t + p_T^2}. \tag{4.1}$$

This integral in the general case requires a UV cutoff. An alternative way to get the final amplitude for the small-angle high-energy scattering is to use the eikonalized form of the amplitude [9, 4, 8]

$$\mathcal{M}_{\rm eik}(s,t) = 2is \int {\rm e}^{i\mathbf{q}\cdot\mathbf{b}} \left(1 - {\rm e}^{i\chi(s,b)}\right) d^2 b, \tag{4.2}$$

where the two-dimensional vectors $\mathbf{q}$, $\mathbf{b}$ lie in the transverse plane, with $\mathbf{b}$ the impact parameter vector. The transverse component $\mathbf{q}$ of the momentum transfer in this approximation satisfies $\mathbf{q}^2 \approx -q^\mu q_\mu$, so that $t \simeq -\mathbf{q}^2$. This expression in the usual four-dimensional theory corresponds to summation of the ladder and crossed-ladder diagrams (for a detailed calculation within the quantized linearized General Relativity see [15]). Actually, this involves UV divergent loop diagrams, but it can be shown that the leading contribution in the high-energy limit is independent on the cutoff. In the ADD linearized gravity the situation is believed to be the same, though no explicit analysis is available. Therefore our classical calculation provides an independent check of the applicability of the eikonal approximation in the ADD framework.

The Born amplitude corresponds to the first term in the expansion of the exponential in (4.2)

$$\mathcal{M}_{\rm Born}(s,t) = 2s \int {\rm e}^{i\mathbf{q}\cdot\mathbf{b}} \chi(s,b) d^2 b, \tag{4.3}$$

and is used to extract the eikonal χ as its inverse Fourier-transform

$$\chi(s,b) = \frac{1}{2s} \int {\rm e}^{-i\mathbf{q}\cdot\mathbf{b}} \mathcal{M}_{\rm Born}(s,t)\, \frac{d^2 q}{(2\pi)^2}. \tag{4.4}$$

Notice, that although the Born amplitude itself may be divergent, the integral (4.4) is finite *if one first integrates over* q*, but not* p_T. Indeed, choose the coordinates as in the previous section to write

$$\chi(s,b) = \frac{s\varkappa_D^2}{4(2\pi)^{\delta+2}} \int {\rm e}^{-iq_x b} \frac{dq_x dq_y dp_T^\delta}{q_x^2 + \varkappa^2}, \quad \varkappa^2 = q_y^2 + p_T^2, \tag{4.5}$$

and integrate first over q_x using a contour integration which gives an exponential factor cutting the potentially divergent integral over p_T. Then, integrate over $q_y = \varkappa \cos\alpha$ and $p_T = \varkappa \sin\alpha$ to obtain

$$\chi(s,b) = \frac{s\varkappa_D^2}{4(2\pi)^{\delta+2}} \int_0^\infty d\varkappa \int_{-\pi/2}^{\pi/2} d\alpha\, \mathrm{e}^{-\varkappa b} S_{\delta-1} \cos^{\delta-1}\alpha\, \varkappa^\delta = \left(\frac{b_c}{b}\right)^\delta , \tag{4.6}$$

where

$$b_c \equiv \frac{1}{\sqrt{\pi}} \left(\frac{\varkappa_D^2 \Gamma(\delta/2)\, s}{16\pi}\right)^{1/\delta} . \tag{4.7}$$

Then, the eikonal amplitude (4.2) becomes

$$\mathcal{M}_{\rm eik}(s,t) = 4\pi i s \int J_0(qb) \left(1 - \mathrm{e}^{i\chi(s,b)}\right) b db. \tag{4.8}$$

The unity in the parenthesis gives no contribution. In the remaining part and in the regime $qb_c \gg 1$ of interest here, one may replace the Bessel function by its asymptotic to obtain

$$\mathcal{M}_{\rm eik}(s,t) = 2\pi i s \int_0^\infty \sqrt{\frac{2b}{\pi q}} \left[\mathrm{e}^{i\psi_+(q,b) - i\pi/4} + \mathrm{e}^{i\psi_-(q,b)+i\pi/4}\right] db, \tag{4.9}$$

where

$$\psi_\pm(q,b) = \pm qb + \left(\frac{b_c}{b}\right)^\delta . \tag{4.10}$$

This can be evaluated using the stationary phase method. The exponent of the second term having no critical points of first order in the domain of integration, is ignored. The stationary point for the first exponent is

$$\left.\frac{d\psi_+}{db}\right|_{b_s} = 0, \quad b_s = \left(\frac{\delta b_c^\delta}{q}\right)^{1/(\delta+1)} , \quad \psi_+''(b_s) = \frac{\delta(\delta+1) b_c^\delta}{b_s^{\delta+2}} , \tag{4.11}$$

and upon integration

$$\mathcal{M}_{\rm eik}(s,t) = \frac{4\pi i s}{\sqrt{\delta(\delta+1)}} \frac{b_s^{(\delta+3)/2}}{b_c^{\delta/2}} \mathrm{e}^{iqb_s} = \frac{4\pi i s \delta^{1/(\delta+1)}}{\sqrt{\delta+1}} \frac{b_c^{\frac{\delta}{\delta+1}}}{q^{\frac{\delta+2}{\delta+1}}} \mathrm{e}^{iqb_s}. \tag{4.12}$$

Substituting b_c from (4.7) one obtains

$$\mathcal{M}_{\rm eik}(s,t) = \frac{4\sqrt{\pi} s \mathrm{e}^{i(qb_s - \pi/2)}}{q\sqrt{\delta+1}} \left(\frac{\varkappa_D^2 s \Gamma(\delta/2+1)}{8\sqrt{\pi} q}\right)^{\frac{1}{\delta+1}} = \frac{4\sqrt{\pi} s \mathrm{e}^{i(qb_s - \pi/2)}}{q\sqrt{\delta+1}} \left(\frac{2\sqrt{\pi} s \Gamma(\delta/2+1)}{M_*^{\delta+2} q}\right)^{\frac{1}{\delta+1}} . \tag{4.13}$$

Note, that although a few intermediate steps are singular for $\delta = 0$ and seem to require a separate discussion, the final formula is valid for $\delta = 0$ as well.

The corresponding cross-section reads

$$\frac{d\sigma_{\rm eik}}{dt} = \frac{1}{16\pi s^2} |\mathcal{M}_{\rm eik}|^2 = \frac{1}{(\delta+1) M_*^2 |t|} \left(4\pi \Gamma^2(1+\delta/2) \frac{s^2}{M_*^2|t|}\right)^{1/(\delta+1)} . \tag{4.14}$$

As advertised, it is identical to our classical result.

5. Conclusions

A purely classical calculation was presented of the high energy elastic scattering cross section in the ADD scenario. Our approach is entirely free of the ambiguities associated with the divergence of the Born amplitude with the virtual graviton exchange typical for ADD. Ultrarelativistic small angle gravitational collision in four dimensions is a special case, in which it agrees with 't Hooft' s result, which in turn coincides with the Born quantum cross-section. In the presence of extra dimensions it was shown that the lowest order small angle classical approximation reproduces the essentially non-perturbative result of the quantum eikonal calculation in the saddle-point approximation. Thus, the classical computation in the above kinematical regime is non-trivial, unambiguous as well as reliable and, therefore, worth applying to other processes like bremsstrahlung [22].

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